1.10. $(x_i)_{i\in I} \subset E$ ~ $(a_i)_{i\in I} \subset \mathbb{R}$.

$A \Rightarrow B$.

$$\left|\sum B_i a_i\right| = \left|\sum B_i \langle f, x_i \rangle\right| \leq \left|\sum B_i \|f\| \|x_i\|\right| = \overset{M.}{\overset{\shortparallel}{\|f\|}} \left|\sum B_i \|x\|\right|$$

$B \Rightarrow A$.

$$g: \quad x \longmapsto f(x)$$

$$\sum_J B_i x_i \longmapsto \sum_J B_i a_i \quad \text{since } x = \sum_J B_i x_i \text{ since } (x_i) \text{ is Hamel Basis.}$$

g is linear by definition.

g is cont since if $\|p-q\| < \delta$, $p = \sum B_i x_i$, $q = \sum C_i x_i$ then

$$\left|f(p) - f(q)\right| = \left|\sum_J B_i a_i - \sum_{J'} C_i a_i\right| = \left|\sum_{J\cup J'} k_i a_i\right| \leq M \left\|\sum_{J\cup J'} B_i - C_i x_i\right\|$$

$$\leq M\delta = \varepsilon.$$

Thus $g \in \bigoplus_{i\in I} \mathbb{R}x_i^{\,*} \subset E$. Hence by Cor 1.2, $\exists f \in E^*$: $f|_{\oplus \mathbb{R}x_i} = g$.

This gives us that $f(x_i) = g(x_i) = a_i$.

1.12. $A \Rightarrow B$ is obvious. $B \Rightarrow A$, using 1.11 $\Rightarrow \left|\sum B_i a_i\right| \leq M \left|\sum B_i f_i\right| = 0 \Rightarrow \sum B_i a_i = 0$.

1.13.

I think they just stress that point (it's own scalar product), because if V has the induced one, then as V is dense in H but is also complete w.r.t the same norm, V=H and the situation becomes trivial.

And yes, indeed the truth is that you will get an isomorphism between V and V*. The whole remark is about the dangers of replacing this canonical isomorphism with the act of actually identifying V and V*. This would give you an inclusion of sets which forces V=H (as sets), which need not be true.